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# Critical exponents, conformal invariance and planar Brownian motion

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**Abstract**

In this review paper, we first discuss some open problems related to two-dimensional self-avoiding paths and critical percolation. We then review some closely related results (joint work with Greg Lawler and Oded Schramm) on critical exponents for two-dimensional simple random walks, Brownian motions and other conformally invariant random objects.

## 1 Introduction

The conjecture that the scaling limits of many two-dimensional systems in statistical physics exhibit conformally invariant behaviour at criticality has led to striking predictions by theoretical physicists concerning, for instance, the values of exponents that describe the behaviour of certian quantities near (or at) the critical temperature. Some of these predictions can be reformulated in elementary terms (see for instance the conjectures for the number of self-avoiding walks of length $n$ on a planar lattice).

From a mathematical perspective, even if the statement of the conjectures are clear, the understanding of these predictions and of the non-rigorous techniques (renormalisation group, conformal field theory, quantum gravity, the link with highest-weight representation of some infinite-dimensional Lie algebras, see e.g. [20, 11]) used by physicists has been limited. Our aim in the present review paper is to present some results derived in joint work with Greg Lawler and Oded Schramm [33, 34, 29, 30, 31, 32] that proves some of these conjectures, and improves substantially our understanding of others. The systems that we will focus on (self-avoiding walks, critical percolation, simple random walks) correspond in the language of conformal field theory to zero central charge.

We structure this paper as follows: In order to put our results into perspective, we start by very briefly describing two models (self-avoiding paths and critical percolation) and some of the conjectures that theoretical physicists have produced and that are, at present, open mathematical problems. Then, we state theorems derived in joint work with Greg Lawler and Oded Schramm [29, 30, 32] concerning critical exponents for simple random walks and planar Brownian motion (these had been also predicted by theoretical physics). We

then show how all these problems are mathematically related, and, in particular, why the geometry of critical percolation in its scaling limit, should be closely related to the geometry of a planar Brownian path via a new increasing set-valued process introduced by Schramm in [38].

# 2 Review of some prediction of theoretical physics

## 2.1 Predictions for self-avoiding walks

We first very briefly describe some predictions of theoretical physics concerning self-avoiding paths in a planar lattice. For a more detailed mathematical account on this subject, see for instance [35].

Consider the square lattice $\mathbb{Z}^2$ and define the set $\Omega_n$ of nearest-neighbour paths of length $n$ started at the origin that are self-avoiding. In other words, $\Omega_n$ is the set of injective functions $\{0,\ldots,n\} \to \mathbb{Z}^2$, such that $w(0) = (0,0) = 0$ and $|w(1) - w(0)| = \cdots = |w(n) - w(n-1)| = 1$.

The first problem is to understand the asymptotic behaviour of the number $a_n := \#\Omega_n$ of such self-avoiding paths when $n \to \infty$. A first trivial observation is that for all $n, m \geq 1$, $a_{n+m} \leq a_n a_m$ because the first $n$ steps and the last $m$ steps of a $n+m$ long self-avoiding path are self-avoiding paths of length $n$ and $m$ respectively. Furthermore, $a_n \geq 2^n$ because if the path goes only upward or to the right, then it is self-avoiding. This leads immediately to the existence of a constant $\mu \in [2,3)$ (called the connectivity constant of the lattice $\mathbb{Z}^2$) such that

$$\mu := \inf_{n \geq 1} (a_n)^{1/n} = \lim_{n \to \infty} (a_n)^{1/n}.$$

Note that if one counts the number $a'_n$ of self-avoiding paths of length $n$ on a triangular lattice, the same argument shows that $(a'_n)^{1/n}$ converges when $n \to \infty$ to some limit $\mu' \geq 3$. The connectivity constant is lattice-dependent.

One can also look at other regular planar lattices, such as the honeycombe lattice. We will say that a property is 'lattice-independent' if it it holds for all these three 'regular' lattices (square lattice, triangular lattice, honeycombe lattice). One possible way to describe a larger class of 'regular' lattices for which our 'lattice-independent' properties should hold, could (but we do not want to discuss this issue in detail here) be that they are transitive (i.e. for each pair of points, there exists a euclidean isometry that maps the lattice onto itself and one of the two points onto the other) and that rescaled simple random walk on this lattice converges to planar Brownian motion. For instance, the lattice $\mathbb{Z} \times 2\mathbb{Z}$ is not allowed.

When $L$ denotes such a planar lattice, we denote by $a_{n,L}$ the number of self-avoiding paths of length $n$ in the lattice starting from a fixed point, and by $\mu_L := \inf_{n \geq 1} (a_{n,L})^{1/n}$ its connectivity constant.

A first striking prediction from theoretical physics is the following:

**Prediction 1 (Nienhuis [37])** *For any regular planar lattice $L$, when $n \to \infty$,*

$$a_n(L) = (\mu_L)^n n^{11/32 + o(1)}.$$

The first important feature is the rational exponent 11/32. The second one is that this result does not depend on the lattice i.e., the first-order term is lattice-dependent while the second is "universal".

The following statement is almost equivalent to the previous prediction. Suppose that we define under the same probability $P_n$ two independent self-avoiding paths $w$ and $w'$ of length $n$ on the lattice $L$ (the law of $w$ and $w'$ is the uniform probability on $\Omega_n$).

**Prediction 2 (Intersection exponents version)** *When $n \to \infty$,*

$$P_n[w\{1, 2, \dots, n\} \cap w'\{0, 1, \dots, n\} = \emptyset] = n^{-11/32+o(1)}.$$

Indeed, $w\{1, 2, \dots, n\}$ and $w'\{0, 1, \dots, n\}$ are disjoint if and only if the concatenation of the two paths $w$ and $w'$ is a self-avoiding path of length $2n$ so that the non-intersection probability is exactly $a_{2n}/(a_n)^2$.

A second question concerns the typical behaviour of a long self-avoiding path, chosen uniformly in $\Omega_n$ when $n$ is large. Let $d(w)$ denote the diameter of $w$. Theoretical physics predicts that the typical diameter is of order $n^{3/4}$. This had already been predicted using a different ('very non-rigorous') argument by Flory [18] in the late 40's. A formal way to describe this prediction is the following:

**Prediction 3 (Nienhuis [37])** *For all $\epsilon > 0$ and all regular lattices, when $n \to \infty$,*

$$P_n\left[d(w) \in [n^{3/4-\epsilon}, n^{3/4+\epsilon}]\right] \to 1.$$

One of the underlying beliefs that lead to these conjectures is that the measure on long self-avoiding paths, suitably rescaled, converges when the length goes to infinity, towards a measure on continuous curves, that posesses some invariance properties under conformal transformations. The counterparts of the previous predictions in terms of this limitting measure then go as follows: Take two independent paths defined under the limitting measure, started at distance $\epsilon$ from each other. Then, the probability that the two paths are disjoint decays like $\epsilon^{11/24}$ when $\epsilon \to 0$. For the second prediction: the Hausdorff dimension of a path defined under the limitting measure is almost surely 4/3.

## 2.2 Predictions for critical planar percolation

We now review some results predicted by theoretical physics concerning critical planar percolation. A more detailed acount on these conjectures for mathematicians can be found for instance in [25]. See [19] for a general introduction to percolation.

Let $p \in (0, 1)$ be fixed. For each edge between neighbouring points of the lattice, erase the edge with probability $1 - p$ and keep it (and call the edge open) with probability $p$ independently for all edges. In other words, for each edge we toss a biased coin to decide whether it is erased or not. This procedure defines a random subgraph of the square grid. It is not difficult to see that the large-scale geometry of this subgraph depends a lot on the value of $p$. In

particular, there exists a critical value $p_c$ (called the critical probability), such that if $p < p_c$, there exists almost surely no unbounded connected component in the random subgraph, while if $p > p_c$, there exists almost surely a unique unbounded connected component of open edges. It is not very difficult to see that the value $p_c$ of the critical probability is lattice-dependent. Kesten has shown that for $L = \mathbb{Z}^2$, $p_c = 1/2$. We are going to be interested in the geometry of large connected component when $p = p_c$.

In regular planar lattices at $p = p_c$ it is known that almost surely no infinite connected component exists. However, a simple duality argument shows that in the square grid, at $p = p_c = 1/2$, for any $n \geq 1$, with probability $1/2$, there exists a path of open edges joining (in that rectangle) the bottom and top boundaries of a fixed $n \times (n+1)$ rectangle. This loosely speaking shows that in a big box, with large probability, there exist connected components of diameter comparable to the size of the box.

Theoretical physics predicts that large-scale properties of the geometry of critical percolation (i.e., percolation on a planar lattice at its critical probability) are lattice-independent (even though the value of the critical probability is lattice-dependent), and, in the scaling limit, invariant under conformal transformations; see e.g. [1, 25]. Furthermore, physicists have produced explicit formulas that describe some of its features.

A first prediction is the following: Consider critical percolation restricted to an $n \times n$ square (in the square lattice, say), and choose the connected component $C$ with largest diameter (among all connected components). The previous observation shows that the diameter of $C$ is of the order of magnitude of $n$. Define the rescaled discrete outer perimeter of $C$, $\partial_n = \partial C/n$, where $\partial C$ is the boundary of the unbounded connected component of the complement of $C$ in the plane.

**Prediction 4 (Cluster boundaries [16, 12, 4])** *The law of $\partial_n$ converges when $n \to \infty$ towards a law $\mu$ on continuous paths $\partial$. Moreover, $\mu$-almost surely: the Hausdorff dimension of $\partial$ is $7/4$, the path $\partial$ is not self-avoiding, and the outer boundary of $\partial$ has Hausdorff dimension $4/3$.*

A weaker version of the first part of this prediction is that for all $\epsilon > 0$, when $n \to \infty$, $P[\#\partial C \in (n^{7/4-\epsilon}, n^{7/4+\epsilon})] \to 1$.

A second prediction concerns the crossing probabilities of a quadrilateral. Suppose that $L > 0$ and $l > 0$, and perform critical percolation in the rectangle $[0, a_n] \times [0, b_n]$ where $a_n$ and $b_n$ are the respective integer parts of $Ln$ and $ln$. Let $x(L, l)$ denote the cross-ratio between the four corners of the $L \times l$ rectangle (more precisely, it is the value $x$ such that there exists a conformal mapping from the rectangle onto the upper half-plane, such that the left and right-hand side of the rectangle are mapped onto the intervals $(-\infty, 0]$ and $[1 - x, 1]$).

**Prediction 5 (Cardy's formula [10])** *When $n \to \infty$, the probability that there exists a path of open edges in the rectangle $[0, a_n] \times [0, b_n]$ joining the*

*left and right-hand sides of the boundary of the rectangle converges to*

$$F(x) = \frac{3\Gamma(2/3)}{\Gamma(1/3)^2} x^{1/3} {}_2F_1(1/3, 2/3, 4/3; x)$$

*where ${}_2F_1$ is the usual hypergeometric function.*

These results are believed to be lattice-independent. This second statement has been predicted by Cardy [9, 10], using and generalising conformal field theory considerations and ideas introduced in [6, 7]. Note that this prediction is of a different nature than the previous ones. It gives an exact formula for an event in the scaling limit rather than just an exponent. Assuming conformal invariance, this prediction can be reformulated in a half-plane as follows:

**Prediction 6 (Cardy's formula in a half-plane)** *For all $a, b > 0$, the probability that there exists a crossing (a path of open edges) joining $(-\infty, -an]$ to $[0, bn]$ in the upper half-plane converges when $n \to \infty$ towards $F(b/(a+b))$.*

Carleson was the first to note that Cardy's formula takes on a very simple form in an equilateral triangle. Suppose $A, B, C$ are the vertices of an equilaterial triangle, say $A = 0, B = e^{2i\pi/3}, C = e^{i\pi/3}$.

**Prediction 7 (Cardy's formula in an equilateral triangle)** *In the scaling limit (performing critical percolation on the grid $\epsilon\mathbb{Z}^2$, say, and letting $\epsilon \to 0$), the law of the left-most point on $[B, C]$ that is connected (in the triangle) to $[A, C]$ is the uniform distribution on $[B, C]$.*

# 3 Brownian exponents

We now come to the core of the present paper and state some of the results derived in the series of papers [29, 30, 31, 32] concerning critical exponents for planar Brownian motions and simple random walks. These are mathematical results (as opposed to the predictions reviewed in the previous section) that had been predicted some 15 or 20 years ago. The proofs of these theorems (that we shall briefly outline in the coming sections) do use conformal invariance, complex analysis and univalent functions.

## 3.1 Intersection exponents

Suppose that $(S_n, n \geq 0)$ and $(S'_n, n \geq 0)$ are two independent simple random walks on the lattice $\mathbb{Z}^2$ that are both started from the origin. We are interested in the asymptotic behaviour (when $n \to \infty$) of the probability that the traces of $S$ and $S'$ are disjoint.

**Theorem 8 ([30])** *When $n \to \infty$,*

$$P[S\{1, 2, \dots, n\} \cap S'\{0, 1, \dots, n\} = \emptyset] = n^{-5/8+o(1)}.$$

This result had been predicted by Duplantier-Kwon [15], see also [13, 14] for another non-rigorous derivation based on quantum gravity ideas and predictions by Khnizhnik, Polyakov and Zamolodchikov.

Note that, as opposed to self-avoiding walks and percolation cluster boundaries, the scaling limit of planar simple random walk is well-understood mathematically: It is planar Brownian motion (this is lattice-independent) and it is invariant under conformal transformations (modulo time-change). This is what makes it possible to prove Theorem 8 as opposed to the analogous prediction for self-avoiding walks. The scaling limit analog of Theorem 8 is the following:

**Theorem 9** *[30] Let $B$ and $B'$ denote two independent planar Brownian motions started at distance 1 from each other. Then, when $t \to \infty$,*

$$P[B[0,t] \cap B'[0,t] = \emptyset] = t^{-5/8+o(1)}.$$

In [29, 30, 31, 32], analogous results concerning non-intersection exponents between more than two Brownian motions (or simple random walks), in the plane or in the half-plane, are derived.

In fact, Theorem 8 is a consequence of Theorem 9 via an invariance principle argument (see [28] and the references therein for the connection between the two results).

## 3.2 Mandelbrot's conjecture

Let $(B_t, t \geq 0)$ denote a planar Brownian motion. Define the hull of $B[0,1]$ as the complement of the unbounded connected component of $\mathbb{C} \setminus B[0,1]$ and define the the outer frontier of $B[0,1]$ as the boundary the hull of $B[0,1]$.

**Theorem 10 ([29, 30, 32])** *Almost surely, the Hausdorff dimension of the outer boundary of $B[0,1]$ is $4/3$.*

This result had been conjectured by Mandelbrot [36] based on simulations and the analogy with the conjectures for self-avoiding walks. See also [14] for a physics approach based on quantum gravity. This theorem is in fact a consequence (using results derived by Lawler in [26]) of the determination of the following critical exponent (called disconnection exponent):

**Theorem 11 ([29, 30, 32])** *If $B$ and $B'$ are two independent planar Brownian motions started from $0$, then, when $t \to \infty$,*

$$P[B[0,t] \cup B'[0,t] \text{ does not disconnect } 1 \text{ from } \infty] = t^{-1/3+o(1)}.$$

Similarly, a consequence of Theorem 9 is that the Hausdorff dimension of the set of cut points of the path $B[0,1]$ is almost surely $3/4$. Analogously, we get that the set of pioneer points (i.e., points $B_t$ that are on the outer boundary of $B[0,t]$) in a planar Brownian path has Hausdorff dimension $7/4$, which - together with Theorem 10 - is reminiscent of Prediction 4. Also, the determination of more general exponents give the multifractal spectrum of the outer frontier of a Brownian path. See [27] and the references therein (papers by Lawler) for the link between critical exponents and Hausdorff dimensions. See also [5].

# 4 Universality

## 4.1 In the plane

A first important step in the proof of Theorems 9 and 10 is the observation that some conformally invariant random probability measures with a special 'restriction' property are identical. Our presentation of universality here differs slightly from that of the preprints [34, 29, 30, 31]. A more extended version of the present approach and of its consequences is in preparation.

More precisely, let $D$ denote the open unit disc, and suppose that $P^0$ is a rotationally invariant probability measure defined on the set of all simply connected compact subsets $K$ of the closed unit disc, that contain the origin and such that $K \cap \partial D$ is just one single point $e(K)$ (we endow this set with a well-chosen $\sigma$-field). Note that the law of $e(K)$ is the uniform probability $\lambda$ on $\partial D$.

For all $x \in D$, the probability measure $P^x$ is defined as the image probability measure of $P^0$ under a Möbius transformation $\Phi$ from $D$ onto $D$ such that $\Phi(0) = x$ (rotational invariance of $P^0$ shows that $P^x$ is independent of the actual choice of $\Phi$). Similarly, for any simply connected open set $\Omega \subset \mathbb{C}$ (that is not identical to the whole plane) and $x \in \Omega$, we define the probability $P^{x,\Omega}$ as the image measure of $P^0$ under a conformal transformation $\Phi$ from $D$ onto $\Omega$ with $\Phi(0) = x$. When $\Phi$ does not extend continuously to the boundary of $D$, one can view $\Phi(K)$ as the union of $\Phi(K \setminus \{e(K)\})$ and the prime end $\Phi(e(K))$.

**Definition 12** *We say that $P^0$ is completely conformally invariant (in short: CCI) if for any simply connected $\Omega \subset D$, for any $x \in D$, the measures $P^{x,D}$ and $P^{x,\Omega}$ are identical when restricted to the family of sets $\{K \ : \ K \cap D \subset \Omega\}$.*

Note that (under $P^{x,\Omega}$), $K \cap D \subset \Omega$ is equivalent to $\Phi(e(K)) \subset \partial D$.

**Theorem 13** *There exists a unique conformally invariant probability measure. It is given by the hull of a Brownian path started from 0 and stopped at its first hitting of the unit circle.*

**Idea of the proof.** First, note that the measure $\tilde{P}$ defined using the hull of a stopped Brownian path is indeed CCI, because of conformal invariance of planar Brownian motion and the strong Markov property. Then, consider another CCI measure $P$. It is sufficient to show that for all simply connected $D' \subset D$ that contains 0,

$$P[K \cap D \subset D'] = \tilde{P}[K \cap D \subset D'] \tag{1}$$

since the family of all such events $\{K \cap D \subset D'\}$ is a generating $\pi$-system of the $\sigma$-field on which we define the measure. Let $\Phi$ denote a conformal map from $D'$ onto $D$ with $\Phi(0) = 0$. Then, because of the CCI property, both sides of (1) are equal to $\lambda(\Phi(\partial D \cap \partial D'))$ and hence equal.

## 4.2 In the half-plane

Analogous (slightly more complicated) arguments can be developped for subsets $K$ of a domain that join two parts of the boundary of this domain (as opposed to joining a point in the interior to the boundary, as in the previous subsection). For convenience, we consider subsets of the equilateral triangle $A = 0$, $B = e^{2i\pi/3}$, $C = e^{i\pi/3}$.

We study probability measures $P$ on the set of all simply connected compact subsets $K$ of $\overline{\mathcal{T}}$ such that

$$K \cap \partial \mathcal{T} = [A, A_1] \cup [A, A_2] \cup \{e(K)\}$$

where $A_1 = A_1(K) \in [A, B]$, $A_2 = A_2(K) \in [A, C]$ and $e(K) \in (B, C)$, and such that $\mathcal{T} \setminus K$ consists of exactly two connected components (having respectively $[A_1, B] \cup [B, e(K)]$ and $[e(K), C] \cup [C, A_2]$ on their boundary).

We say that $P$ satisfies Cardy's formula if the law of $e(K)$ is the uniform probability measure on $[B, C]$.

We say that $P$ satisfies Property I, if for all non-empty $(B', C') \subset [B, C]$, if $\Phi$ denotes the conformal map from $\mathcal{T}$ onto $\mathcal{T}$ with $\Phi(A) = A$, $\Phi(B') = B$ and $\Phi(C') = C$, the image measure of $P$ (restricted to $\{e(K) \in (B', C')\}$) under the mapping $\Phi$ is exactly the measure $P$ restricted to $\{A_1 \in [A, \Phi(B)]\} \cap \{A_2 \in [A, \Phi(C)]\}$.

Suppose that $K'$ is a simply connected compact subset of $\overline{\mathcal{T}}$ such that $K' \cap [B, C] \neq \emptyset$, $K' \cap ([A, B] \cup [A, C]) = \emptyset$, and $\mathcal{T}' := \mathcal{T} \setminus K'$ is simply connected. Let $\Phi$ denote the conformal mapping from $\mathcal{T}'$ onto $\mathcal{T}$ with $\Phi(A) = A$, $\Phi(B) = B$ and $\Phi(C) = C$. We say that $P$ satisfies Property II, if for all such $K'$, the image of $P$ (restricted to $\{K \ : \ K \cap K' = \emptyset\}$) under $\Phi$ is identical to the measure $P$ restricted to the set $\{K \ : \ K \cap \Phi(\partial \mathcal{T}' \setminus \partial \mathcal{T}) = \emptyset\}$.

We the say that $P$ is invariant under restriction if it satisfies Property I and Property II. There are other equivalent definitions and formulations of this restriction property.

**Theorem 14** *There exists a unique probability measure $P$ that satisfies Cardy's formula and is invariant under restriction. It is given by the hull in $\mathcal{T}$ (see the precise definition below) of Brownian motion, started from $A$, reflected with oblique angle $\pi/3$ (pointing 'away' from $A$) on [A,B] and $[A, C]$, and stopped when reaching $[B, C]$.*

Here and in the sequel, the hull in the upper half-plane (or $H$-hull) of a compact set $K \subset \overline{H}$ with $0 \in K$ is the complement of the unbounded connected component of $\overline{H} \setminus K$. The hulll in $\mathcal{T}$, or $\mathcal{T}$-hull, of a compact subset of $\overline{\mathcal{T}}$ that contains $A$, is the complement of the union of the connected components of $\overline{\mathcal{T}} \setminus K$ that have $B$ or $C$ on their boundaries.

**Idea of the proof.** Uniqueness follows easily from a similar argument as in Theorem 13, by identifying $P(K \cap \mathcal{T} \subset \mathcal{T}')$ for a certain class of simply connected subsets $\mathcal{T}'$ of $\mathcal{T}$. For existence, one needs to verify that such a reflected Brownian motion satisfies Cardy's formula, and the restriction property

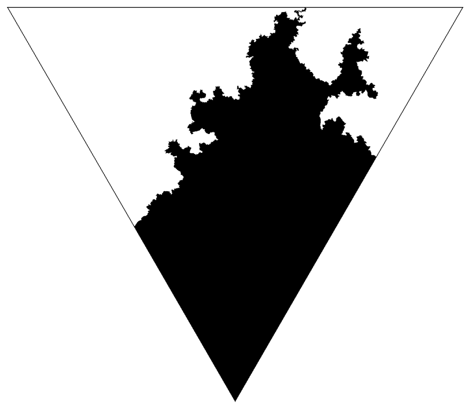

Figure 1: The $\mathcal{T}$-hull of reflected Brownian motion

follows from conformal invariance and the strong Markov property for such a reflected Brownian motion.

Let us briefly indicate how this reflected Brownian motion is defined (see for instance [40] for details). We first define it in the upper half-plane. Define for any $x \in \mathbb{R}$, the vector $u(x) = \exp(i\pi/3)$ if $x \geq 0$ and $u(x) = \exp(2i\pi/3)$ if $x < 0$. Suppose that $B(t)$ is an ordinary planar Brownian path started from 0. Then, there exists a unique pair $(Z_t, \ell_t)$ of continuous processes such that $Z_t$ takes its values in $\overline{H}$, $\ell_t$ is a non-decreasing real-valued function with $\ell_0 = 0$ that increases only when $Z_t \in \mathbb{R}$, and

$$Z_t = B_t + \int_0^t u(Z_s) d\ell_s.$$

The process $(Z_t, t \geq 0)$ is called the reflected Brownian motion in $H$ with reflection vector field $u(\cdot)$. At each time $t$, we define $V_t$ as the $H$-hull of $Z[0,t]$. In order to define the $\mathcal{T}$-hull of reflected Brownian motion in the triangle that is refered to in the Theorem, consider for instance a conformal mapping $\Phi$ from $H$ onto the triangle, such that $\Phi(0) = A$, $\Phi(\infty) = B$, $\Phi(M) = C$ for some real $M > 0$. Then the $\mathcal{T}$-hull is $\Phi(V_T)$ where $T$ is the first time at which $Z$ hits $[M, \infty]$.

# 5 Schramm's processes

## 5.1 In the plane

It is possible to contruct a CCI measure $P$ using Loewner's differential equation (that encodes a certain class of growing families of compact sets) driven by a Brownian motion. See, for instance, [17] for a general introduction to Loewner's equation.

For any simply connected compact $K$ such that the complement of $K$ in the plane is conformally equivalent to the complement of a disc, Riemann's mapping theorem shows that there exists a unique $\alpha_K \in \mathbb{R}$ and a unique conformal map $\hat{f}_K$ that maps the complement of $K$ onto the complement of the unit disk in such a way that

$$\hat{f}_K(z) = ze^{-\alpha_K} + O(1)$$

when $z \to \infty$.

Suppose now that $(\zeta(t), t \in \mathbb{R})$ is a continuous function taking values on the unit circle. For any $z \in \mathbb{C}$, define $f_t(z)$ as the solution of the ordinary differential equation

$$\partial_t f_t(z) = -f_t(z) \frac{f_t(z) + \zeta(t)}{f_t(z) - \zeta(t)}$$

such that

$$\lim_{t \to -\infty} e^{-t} f_t(z) = z$$

For any fixed $z \neq 0$, the mapping $t \mapsto f_t(z)$ is well-defined up to a possibly infinite 'explosion time' $T_z$, at which $f_t(z)$ hits the singularity $\zeta(t)$ (and we put $T_0 = -\infty$). Simple considerations show that for any time $t \in \mathbb{R}$, $\alpha_{K_t} = t$, and $f_t = \hat{f}_{K_t}$, where

$$K_t = \{z \in \mathbb{C} \ : \ T_z \leq t\}$$

i.e., $f_t$ is the conformal mapping from the complement of $K_t$ onto the complement of the unit disc such that $f_t(z) = ze^{-t} + o(z)$ when $z \to \infty$.

We now take $\zeta(t) = \exp(i\sqrt{\kappa}W_t), t \in \mathbb{R}$, where $\kappa > 0$ is a fixed constant and $(W_t, t \in \mathbb{R})$ denotes a one-dimensional Brownian motion such that the law of $\sqrt{\kappa}W(0)$ is the uniform distribution on $[0, 2\pi]$. We call $(K_t, t \in \mathbb{R})$ Schramm's radial process with parameter $\kappa$ (in [38, 30], it is refered to as $SLE_\kappa$ for Stochastic Loewner Evolution process). Then, let $T$ denote the first time at which $K_t$ intersects the unit circle. The set $K_T$ is a (random) simply connected compact set that intersects the unit circle at just one point.

**Theorem 15** *[29, 30] If $\kappa = 6$, the law of $K_T$ is a CCI probability measure.*

In fact, a more general result (complete conformal invariance of $SLE_6$ as a process) also holds, see [29, 30]. This theorem may seem quite surprising. Note that it fails for all other values of $\kappa$. The idea of the proof is to prove 'invariance' of the law under infinitesimal deformations of $D$.

A direct consequence of Theorems 15 and 13 is that the law of $K_T$ is identical to the law of the hull of a stopped Brownian path. This is a rather surprising result since the two processes (hulls of planar Brownian motion and $SLE_6$) are a priori very different. For instance, the joint distribution at the first hitting times of the circles of radius 1 and 2 are not the same for $SLE_6$ and for the hull of a planar Brownian path.

## 5.2 In the half-plane

Similarly, one can construct natural processes of compact subsets of the closed upper-half plane $\overline{H} = \{z \ : \ \Im(z) \geq 0\}$ that are 'growing from the boundary'.

For any simply connected compact subset $K$ of $\overline{H}$ such that $0 \in K$ and $H \setminus K$ is simply connected, there exists a unique number $\beta_K \geq 0$ and a unique conformal map $\hat{g}_K$ from $H \setminus K$ onto $H$ such that $\hat{g}_K(\infty) = \infty$ and

$$\hat{g}_K(z) = z + \frac{\beta_K}{z} + o\left(\frac{1}{z}\right)$$

when $z \to \infty$. $\beta_K$ which is increasing in $K$, is a half-space analog of capacity.

Suppose now that $(a(t), t \geq 0)$ is a continuous real-valued function, and define for all $z \in \overline{H}$, the solution $g_t(z)$ to the ordinary differential equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - a(t)}$$

with $g_0(z) = z$. This equation is well-defined up to the (possibly infinite) time $T_z$ at which $g_t(z)$ hits $a(t)$. Then, define

$$K_t := \{z \in \overline{H} \ : \ T_z \leq t\}.$$

$(K_t, t \geq 0)$ is an increasing family of compact sets, and it is not difficult to see that for each time $t \geq 0$, $\beta_{K_t} = 2t$ and $g_t = \hat{g}_{K_t}$ i.e., $g_t$ is the unique conformal mapping from $H \setminus K_t$ onto $H$ such that $g_t(\infty) = \infty$, and $g_t(z) = z + 2t/z + o(1/z)$ when $z \to \infty$.

When $a(t) = \sqrt{\kappa} W_t$, where $W$ is one-dimensional Brownian motion with $W(0) = 0$, we get a random increasing family $(K_t, t \geq 0)$ that we call Schramm's chordal process with parameter $\kappa$ (it is refered to as chordal $SLE_\kappa$ in [38, 29, 30, 31]).

Note that if we put $Z_t(z) = W_t - \kappa^{-1/2} g_t(z)$, then

$$Z_t(z) = W_t + \int_0^t \frac{2}{\kappa Z_s(z)} ds$$

so that $Z$ can be interpreted as a (translation of a) complex Bessel flow of dimension $1 + (4/\kappa)$. If $\kappa < 4$, it is easy to see, by comparison with a Bessel process, that almost surely $T_z < \infty$ for all $z \in \overline{H}$, in other words, $\cup_{t \geq 0} K_t = \overline{H}$.

The scaling properties of Brownian motion easily show that it is possible to define (modulo increasing time-reparametrization) the law of an increasing process of hulls in any simply connected set. More precisely, if $\Phi$ is a conformal mapping from $H$ onto some simply connected domain $\Omega$, then we say that $(\Phi(K_t), t \geq 0)$ is a chordal Schramm process (with parameter $\kappa$) started from $\Phi(0)$ aiming at $\Phi(\infty)$ in $\Omega$. This process is well-defined modulo increasing time-reparametrization (i.e., if there exists a (random) continuous increasing $\psi$ such that for all $t$, $K_t = K'_{\psi(t)}$, then we say that the two processes $K$ and $K'$ are equal).

For the remainder of this paper, we will assume that $\kappa = 6$ as this case exhibits many very interesting properties. It is possible to compute explicitely certain probabilities. For instance, if $a, b > 0$,

$$P(T_{-a} < T_b) = F(b/(a+b))$$

where $F$ is the same as in Cardy's formula In fact, much more is true:

**Theorem 16 ([29])** *Consider $SLE_6$ in the equilateral triangle, started from $A$ and aiming at some point $M \in [B,C]$. Let $T$ denote the first time $t$ at which $K_t \cap [B,C] \neq \emptyset$. Then, the law of $K_{T-} = \overline{\cup_{t<T} K_t}$ is independent of the choice of $M \in [B,C]$, it satisfies Cardy's formula and it is invariant under restriction.*

Again, this property is only valid when $\kappa = 6$ and there exists a more general version in terms of processes [29].

## 5.3 Link with reflected Brownian motion

From Theorem 16, Theorem 14 and the strong Markov property, it follows that the chordal process $(K_t, t \geq 0)$ can be reinterpreted in terms of reflected Brownian motions. In particular, the law of $K_{T-}$ is that of the $\mathcal{T}$-hull of a stopped Brownian motion with oblique reflection. More generally, finite-dimensional marginals of the chordal $SLE_6$ process can for instance be constructed as follows (other more general statements with other stopping times hold as well).

Suppose that $J_1, \ldots, J_p$ is a decreasing family of closed subsets of $\overline{H}$ such that $H \setminus J_1, \ldots, H \setminus J_p$ are simply connected. When $(K_t, t \geq 0)$ is chordal $SLE_6$ in $\overline{H}$ (started at 0 and aiming at infinity), we put, for all $j \leq p$,

$$T_j = \inf\{t > 0 \ : \ K_t \cap J_j \neq \emptyset\}.$$

Let $V$ denote a simply connected compact subset of the upper half-plane such that $H' = H \setminus V$ is simply connected and let $x \in \partial H'$. Define a reflected Brownian motion $(B_s, s \geq 0)$ in $H'$ with oblique reflection angle (angle $\pi/3$ on the part of the boundary between $x$ and $+\infty$ and reflection $2\pi/3$ between $-\infty$ and $x$) as the conformal image in $H'$ of reflected Brownian motion in $H$. If $J$ is a compact set, define the stopping time $S = S(J)$ at which $V \cup B[0,S]$ intersects $J$ for the first time, and the hull $\mathcal{V} = \mathcal{V}(V, x, J)$ of $V \cup B[0, S(J)]$ in $H$.

Now, define recursively (using each time independent Brownian motions), $V_0 = \{0\}$, $x_0 = 0$,

$$V_{j+1} = \mathcal{V}(V_j, x_j, J_j) \text{ and } x_{j+1} = B(S(J_j)).$$

**Theorem 17** *The laws of $(V_1, \ldots, V_p)$ and of $(K_{T_1}, \ldots, K_{T_p})$ are identical.*

In particular, this implies that the law of $V_p$ is identical to that of $K_{T_p}$ and therefore is independent of $p$ and $J_1, \ldots, J_{p-1}$. This can be viewed as a way to reformulate the restriction property for $SLE_6$ in terms of reflected Brownian motions only. So far, there is no direct proof of this fact that does not use the link with Schramm's process.

### 5.4 Relation between radial and chordal $SLE_6$

Radial and chordal Schramm processes with parameter 6 are very closely related. The CCI property and the restriction property in their 'process' versions can be very non-rigorously described as follows: The evolution of $K_t$ (for radial and chordal $SLE_6$) depends on $K_t$ only in a local way i.e., suppose that at time $t_0$, $K_t$ is increasing near some point $x \in \partial K_{t_0}$, then the evolution of $K_t$ immediately after $t_0$ depends only on how $K_{t_0}$ looks like in the neighbourhood of $x$ (for a rigorous version of this statement, see [29, 30]). The link between radial and chordal $SLE_6$ (and this is only true when $\kappa = 6$) is that this local evolution is the same for radial and chordal $SLE_6$, see [30]. This also leads to a description of the finite-dimensional marginal laws of radial $SLE_6$ in terms of hulls of reflected Brownian motion.

# 6 Computation of the exponents

We now give a very brief outline of the proof of Theorem 8. As a consequence of the results stated in the last two sections, hulls of Brownian paths at certain stopping times can be also constructed in an a priori completely different way using $SLE_6$. The latter turns out to be much better suited to compute probabilities involving only the shape of its complement, as $SLE_6$ is a process that is 'continuously growing to the outside', whereas Brownian motion does incursions inside its own hull, so that for instance the point at which the Brownian hull 'grows' makes a lot of jumps.

It is very easy to show that Theorem 8 is a consequence of the following fact that we shall now derive: Let $B$ and $B'$ denote two independent planar Brownian motions started from 0 and $\epsilon > 0$, and killed when they hit the unit circle. Then, when $\epsilon \to 0$,

$$P[B \cap B' = \emptyset] = \epsilon^{5/4+o(1)}. \tag{2}$$

To derive (2), note first that if $h_{K(B)}$ denotes the harmonic measure of $\partial D$ at $\epsilon$ in $D \setminus K(B)$ (here $K(B)$ is the hull of $B$), then

$$P[B \cap B' = \emptyset | B] = h_{K(B)}.$$

Hence, using Theorem 13, shows that

$$P[B \cap B' = \emptyset] = E[h_{K(B)}] = E[h_{K_T}]$$

where $K_T$ is radial $SLE_6$ stopped as in Theorem 15. The construction of $K_T$ via Loewner's equation makes it possible to study explicitly the asymptotic behaviour of $E[h_{K_T}]$ when $\epsilon \to 0$ by computing the highest eigenvalue computation of a differential operator, see [30]. From this, (2) follows.

Similar (though more involved) arguments lead to Theorem 10 and to the determination of many other such critical exponents that are defined in terms of planar Brownian paths [29, 30, 31, 32].

# 7 The conjecture for percolation

Suppose that one performs critical bond percolation ($p = p_c = 1/2$) in the discrete half-plane $\mathbb{Z} \times \mathbb{N}$. Decide that all edges of the type $[x, x+1]$ when $x$ is on the real axis are erased when $x \geq 0$ and open when $x < 0$. Then, there exists a unique infinite cluster $C_-$ formed by the negative half-axis and the union of all clusters that are attached to it. We now explore the outer boundary of this cluster, starting from the point $(0, 0)$. In other words, we follow the left-most possible path that is not allowed to cross open edges as shown in the picture below (for convenience, we draw the line that stays at 'distance' 1/4 from $C_-$). The open edges are the thin plain lines, and the exploration process is the thick plain line.

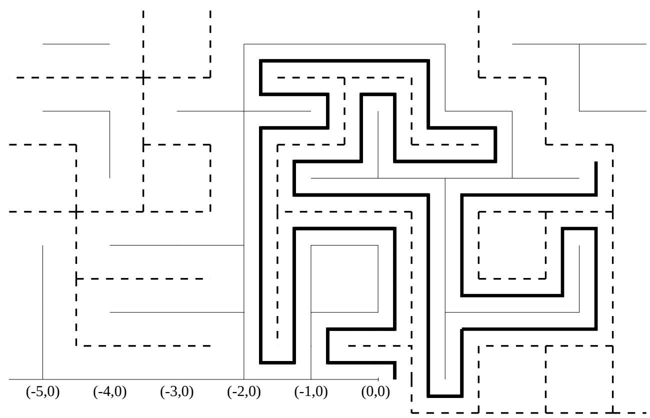


Figure 2: The discrete exploration process

Note that this exploration process is almost symmetric because of the self-duality property of the planar lattice: It is (almost) the same than exploring the outer boundary of the cluster of 'open' edges in the dual lattice (i.e. duals of closed edges in the original lattice - the thick dashed edges in the picture) attached to the 'positive half-line' $(1-i)/2 + \mathbb{N}$ (other choices of the lattice make this exploration process perfectly symmetric).

The conformal invariance conjecture leads naturally to the conjecture [38, 39] that in the scaling limit, this exploration process can be described using Loewner's differential equation in the upper half-plane, and that the driving process $a(t)$ is a continuous symmetric Markov process with stationary increments i.e. $a(t) = \sqrt{\kappa} W_t$ for some parameter $\kappa$. It is then easy to identify $\kappa = 6$ as the only possible candidate (looking for instance at the probability of crossing a square, see [39]), and the restriction property gives additional support to this conjecture. In particular, note that Cardy's formula for Schramm's process with parameter 6 would indeed correspond to the crossing probability.

So far, there is no mathematical proof of the fact that this exploration process converges to Schramm's process. However, Theorem 17 and the fact that the

exploration process can be viewed as a discrete random walk reflected on its past hull gives at least a heuristic hand-waving justification.

One can also recover non-rigorously the exponents predicted for self-avoiding walks (see e.g. [34]) using the exponents derived (rigorously) for Brownian motions and Schramm processes.

The only discrete models that have been mathematically shown to exhibit some conformal invariance properties in the scaling limit - apart from simple random walks - are those studied by Kenyon in [21, 22, 23, 24], namely loop-erased walks and uniform spanning trees. These are conjectured to correspond to Schramm's processes with parameters 2 and 8, see [38]. For partial results concerning percolation scaling limit and its conformal invariance, see [2, 3, 8].

For some other critical exponents related to Hausdorff dimensions of conformally invariant exceptional subsets of the planar Brownian curve (such as pivoting cut points for instance) that we do not know (yet?) the value of, see [5].

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